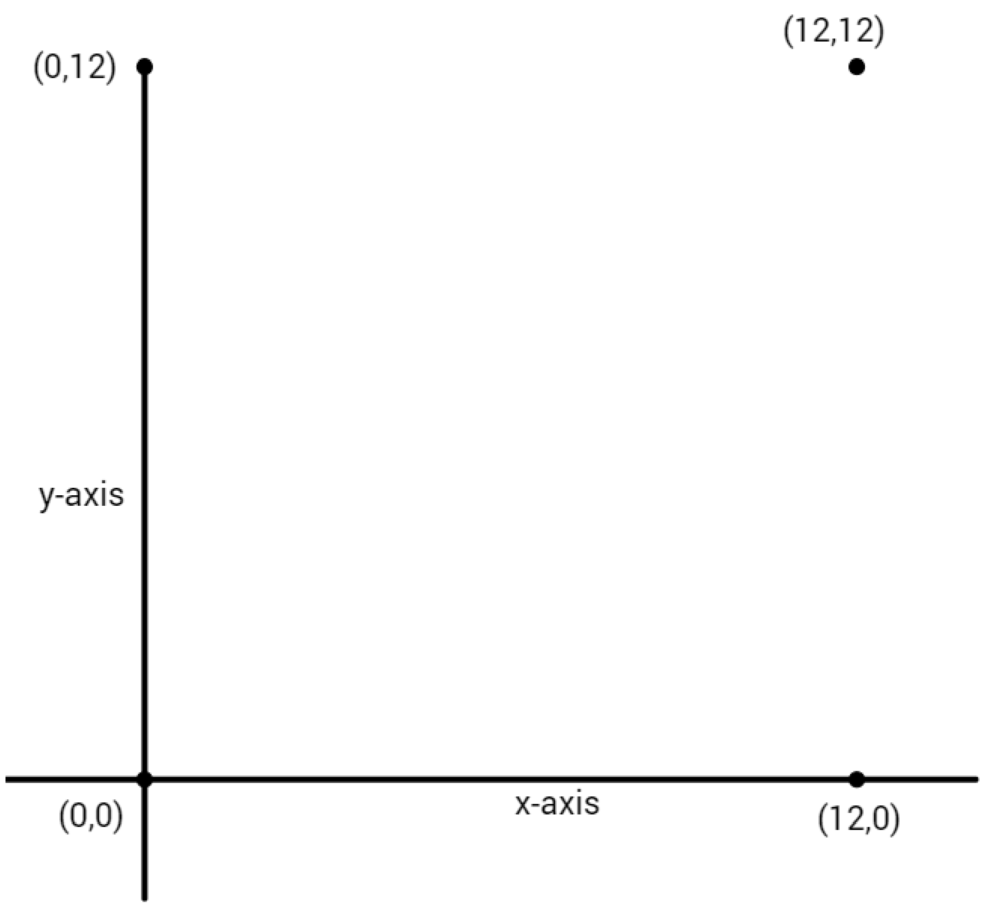
2021 workshop presentation of 20210417 (version of 20210408)

Part 2: Use of two dimensional geometry

Basic definitions:
Definition 1:
There are two axes in the xy-plane, named the $x - axis$ and the $y - axis$. In the figure below these axes are two lines crossing each other at $90°$. The point of intersection is $(0, 0)$.

Definition 2: In the figure below we show some points using the notation $P = (x, y)$ in the xy-plane where x is the x-coordinate and y is the y-coordinate of the point P.



Quiz 1:
Suppose that the point $\boldsymbol{P = (a, b)}$ where $\boldsymbol{a = 2b}$ and $\boldsymbol{a + 2b = 40}$. What is $\boldsymbol{(a, b)}$?

1. $\boldsymbol{(5, 10)}$
2. $\boldsymbol{(10, 5)}$
3. $\boldsymbol{(10, 20)}$
4. $\boldsymbol{(20, 10)}$
5. $\boldsymbol{(15, 15)}$

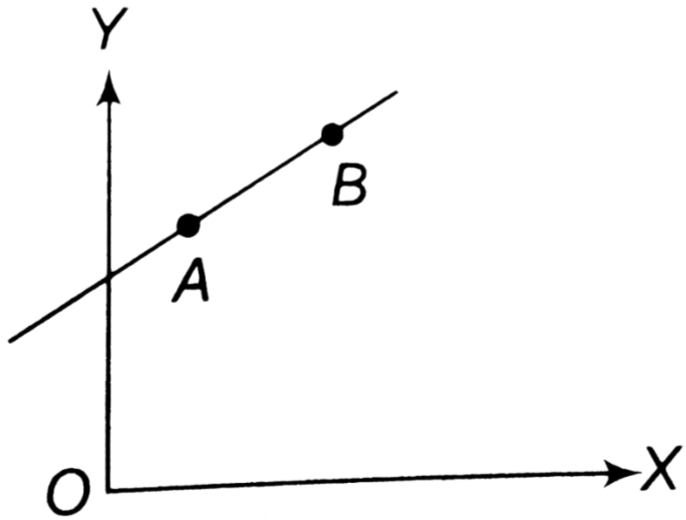
Answer to Quiz 1:
$\mathbf{2b + 2b = 40, 4b = 40, b = 10, a = 2b = 20.}$
Correct answer: $(\mathbf{20}, \mathbf{10})$**.**

Examples of the use of xy-plane figures:
a. Maps identifying the locations of places in a city where the x-coordinate indicates the West-East direction and the y-coordinate indicates the South-North direction.
b. Yearly prices of a specific product where the x-coordinate is the year and the y-coordinate is price in $.

Definition 3: the slope of a line segment in the xy-plane is defined as the rate of change in the y direction divided by the rate of change in the x direction.

Example: if $P = (u, v)$ and $Q = (t, w)$ then the slope of the line segment from P to Q is:
$m = \frac{w-v}{t-u}$ **.**

In the figure below there is a line that goes through 2 points: $A = (2,7)$ and $B = (7,10)$.



Quiz 2:
What is the slope of a line segment from the point $A = (2,7)$ to the point $B = (7,10)$?

1. $\frac{2}{3}$

2. $\frac{3}{4}$

3. $\frac{3}{5}$

4. $\frac{4}{5}$

5. $\frac{5}{7}$

Answer to Quiz 2:
$\frac{10-7}{7-2} = \frac{3}{5}$.

Quiz 3:
The slope of a line segment from the point $(3, 5)$ to the point $(7, y)$ is $\frac{5}{2}$. What is the value of y?
1. 9
2. 12
3. 15
4. 21
5. 25

Answer to Quiz 3:
The slope satisfies the following equation: $\frac{y-5}{7-3} = \frac{5}{2}$**.**
Thus, $\frac{y-5}{4} = \frac{5}{2}$**,** $y - 5 = \frac{4\times 5}{2} = 10$**, so,** $y = 15$**.**

Quiz 4:
The slope of a line segment from the point $(3, 5)$ **to the point** $(x, x+8)$ **is** $\frac{5}{2}$**. What is the value of** x**?**

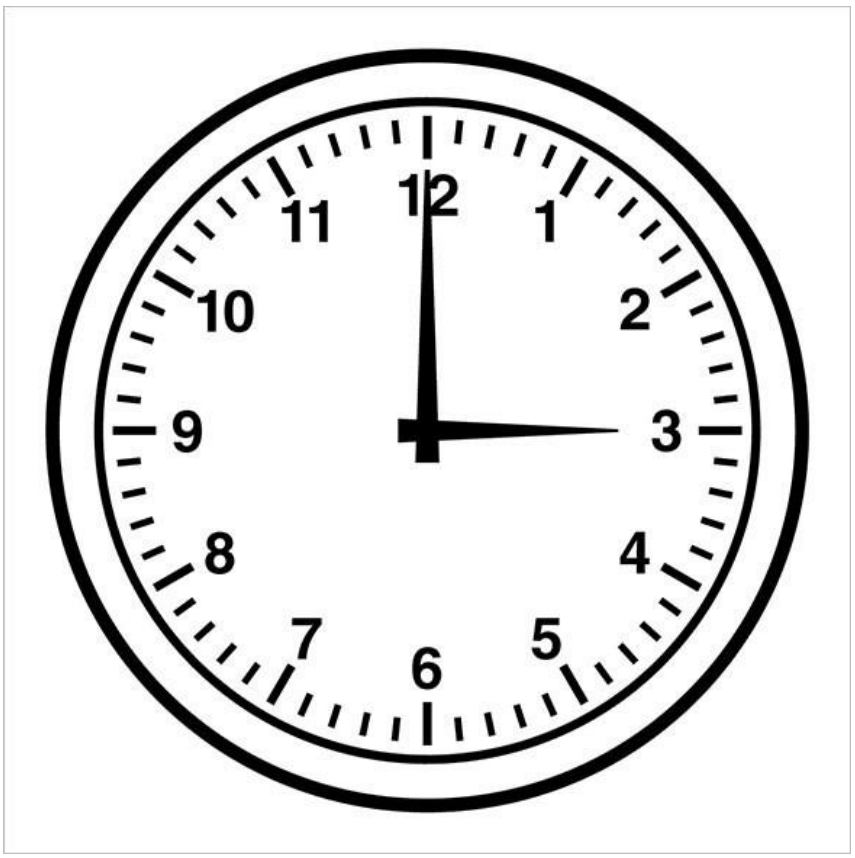
1. **3**
2. **7**
3. **10**
4. **12**
5. **15**

Answer to Quiz 4:

The slope satisfies the following equation: $\frac{x+8-5}{x-3} = \frac{5}{2}$.

Thus, $\frac{x+3}{x-3} = \frac{5}{2}$, $2x + 6 = 5x - 15$, $21 = 3x$, so $x = 7$.

The clock below has two hands: an hour hand (H-hand) that completes one full circle ($360°$) every 12 hours, and a minute hand (M-hand) that completes one full circle every hour. The figure shows the time of $03:00$ (3 hours and 00 minutes) when the H-hand points towards the number 3 and the M-hand points towards the number 12. At this time, the angle between the two hands of the clock is $90°$.



Quiz 5a:
How many degrees does the H-hand travel in 24 minutes?
1. **2**
2. **6**
3. **9**
4. **12**
5. **15**

Quiz 5b:
How many degrees does the M-hand travel in 24 minutes?
1. **96**
2. **112**
3. **144**
4. **150**
5. **172**

Answer to Quiz 5a:

$\frac{360}{12} = 30$, **so the H-hand travels** $30°$ **every hour.**

$30 \times \frac{24}{60} = \frac{24}{2} = 12$, **so in 24 minutes it travels** $12°$**.**

Answer to Quiz 5b:

The M-hand travels $360°$ **every hour.**

$360 \times \frac{24}{60} = 6 \times 24 = 144$, **so in 24 minutes it travels** $144°$**.**

Definition of the time as a single number between 0 and 12:

Define the time t, **as a positive number,** $0 \leq t < 12$. **For example: the time** $t = 0$ **corresponds to the time of** $12:00$ **in the 4-digits notation.**

Examples:

1) At time $t = 0$ **both hands point upwards in the direction of the number 12 in a clock figure similar to the previous figure.**

2) at time $t = 3$, **the H-hand points in the direction of the number 3, and the M-hand points in the direction of the number 12.**

3) at time $t = 3.5 = \frac{7}{2}$, **(or** $03:30$ **in 4 digits notation), the H-hand points in a direction which is halfway between the numbers 3 and 4, and the M-hand points in the direction of the number 6.**

As in example 3, in a fraction notation, in lowest terms, the fraction $\frac{7}{2}$ **is the time** $t = 3.5$ **in decimal notation.**

Quiz 6:

What fraction, in lowest terms, represents the time $07:45$?

1. $\frac{9}{7}$

2. $\frac{52}{12}$

3. 7.9

4. $\frac{31}{4}$

5. $\frac{124}{12}$

Answer to Quiz 6:

The time of $07:45$ is $7+\frac{45}{60}=7+\frac{3}{4}$ or $\frac{7\times 4+3}{4}=\frac{31}{4}$.

Quiz 7a:

What is the first time after $12:00$ (4-digits notation), when both hands of the clock point in the same direction? Express your answer as a fraction in lowest terms.

1. $\frac{25}{24}$

2. $\frac{13}{12}$

3. $\frac{12}{11}$

4. $\frac{11}{10}$

5. $\frac{7}{6}$

Quiz 7b:

How many times during a 12 hour period do both hands of the clock point in the same direction?

1. 6

2. 11

3. 12

4. 22

5. 24

Answer to Quiz 7a:

For each full circle of the M-hand, the H-hand travels $\frac{1}{12}$ circle. Thus, the circular speed of the M-hand is 12 times the circular speed of the H-hand. Obviously, the first time after $12:00$ where both hands point at the same direction is some time shortly after $01:05$.

Define that time as x.

a) The time that the H-hand traveled after $01:00$ is $x - 1$.

b) The time that the M-hand traveled after $01:00$ is x.

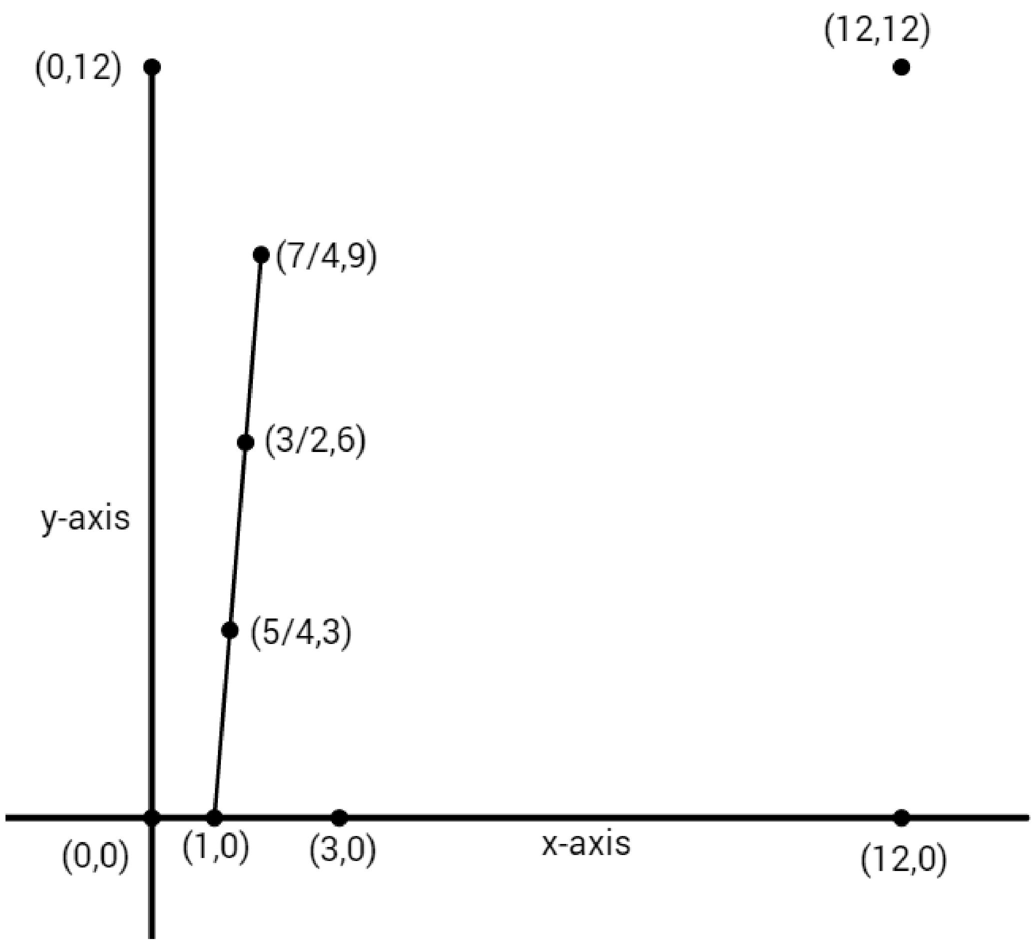
Since the M-hand travels 12 times faster than the H-hand, x satisfies the following equation: $(x - 1) \times 12 = x$.

So, $12x - 12 = x$, $11x = 12$, so, $x = \frac{12}{11}$.

Answer to Quiz 7b:

Overlapping occurs at 12 and then once at some time after every hour except after the hour of 11. So, the total number is 11.

Define a different graphic representation of time.
The 12×12 square (the figure below) where the $x - axis$ is the location of the H-hand, $0 \leq x < 12$, and the $y - axis$ is the location of the M-hand, $0 \leq y < 12$.
Not every point in this figure represents a valid time. The following valid times are shown on this graph using 3 different notations each: $t_1 = 01{:}00 = 1 = (1, 0)$,
$t_2 = 01{:}15 = \frac{5}{4} = (\frac{5}{4}, 3)$, $t_3 = 01{:}30 = \frac{3}{2} = (\frac{3}{2}, 6)$,
$t_4 = 01{:}45 = \frac{7}{4} = (\frac{7}{4}, 9)$, and $t_5 = 03{:}00 = 3 = (3, 0)$.
All points of valid times are located on the straight line segments. (So the line from t_1 to t_4 represent valid times).



Quiz 8:
What is the slope of the line segment from t_1 to t_4 ?

1. 1

2. 6

3. 12

4. 24

5. 60

Answer to Quiz 8:
Using the two end points $(\mathbf{1},\mathbf{0})$ and $(\frac{7}{4},\mathbf{9})$ of the line segment, we get: $\frac{9-0}{\frac{7}{4}-1}=\frac{36}{7-4}=\frac{36}{3}=\mathbf{12}.$

Quiz 9:
Extend the line in the diagram until $y=12$. What is the length of the extended line?

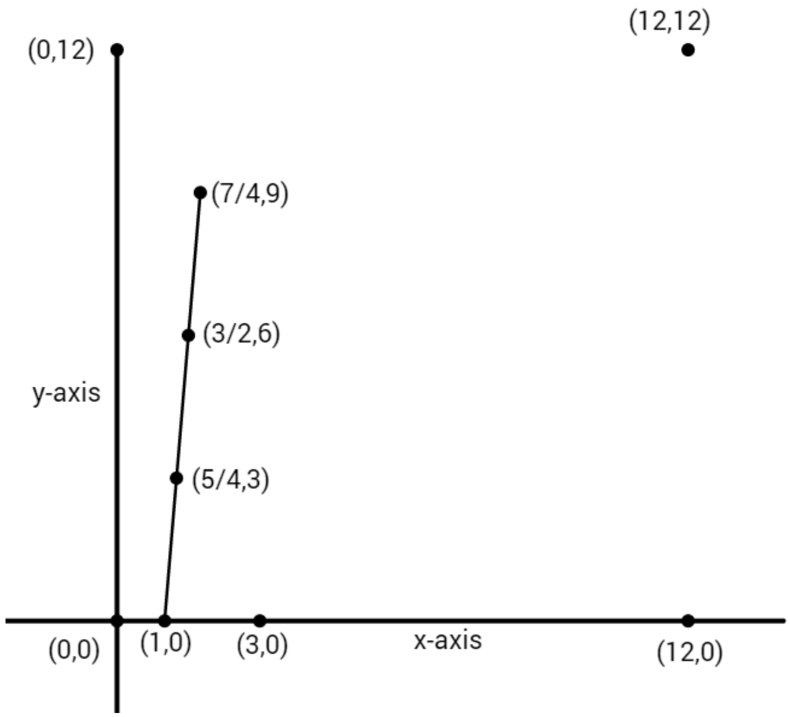
1. $\sqrt{13}$

2. 13

3. $\sqrt{72}$

4. $\sqrt{145}$

5. 145



Answer to Quiz 9:

A triangle (not shown in the figure) with vertices at $(1, 0)$, $(2, 0)$, and $(2, 12)$ is a right triangle. Use the Pythagorean Theorem, $a^2 + b^2 = c^2$ for a right triangle, with $a = 1$, and $b = 12$. Thus, $c^2 = 1^2 + 12^2 = 1 + 144 = 145$. So, $c = \sqrt{145}$.

In conclusion: all valid times are located on straight lines, each with slope of 12, and each starting at one of the following points: $(0, 0), (1, 0), (2, 0), \cdots, (11, 0)$, or a total of 12 parallel straight lines (each with slope of 12).

Quiz 10:
In the figure below, what is the largest x-coordinate of a valid time whose y-coordinate is 8? Express the answer as a fraction in lowest terms.

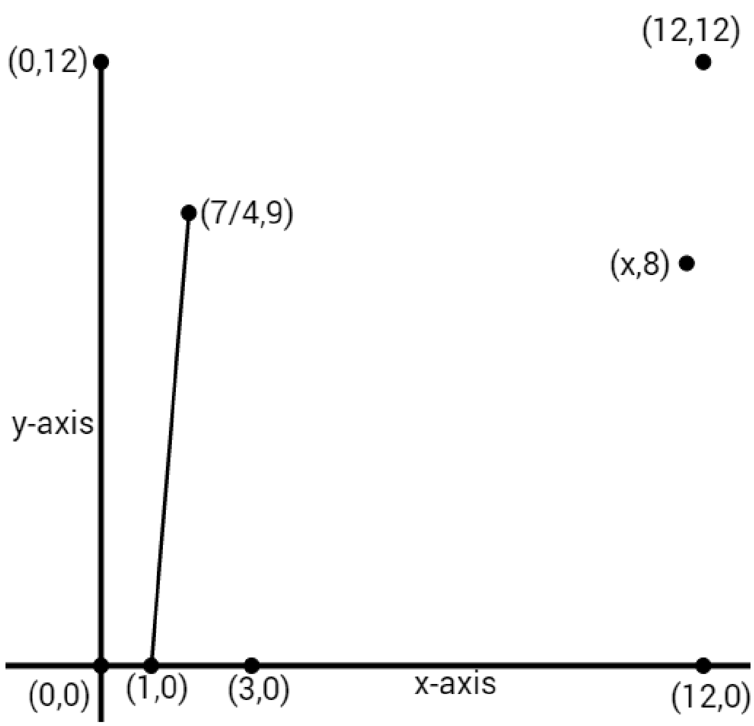
1. $\frac{8}{11}$

2. $\frac{40}{11}$

3. $\frac{47}{12}$

4. $\frac{35}{3}$

5. $\frac{63}{5}$



Answer to Quiz 10:

In the notation that we use, any valid time $0 \le x < 12$, can be rewritten as x= $h + \frac{r}{5}$, where $h = 0, 1, 2, \cdots, 11$. h is the hour (integer value), and the minute value (not necessarily integer) is r, $0 \le r < 60$. Thus, the y-coordinate of this time is $y = \frac{r}{5}$. So, given that $8 = \frac{r}{5}$, $r = 5 \times 8 = 40$. Thus, the x-coordinate, satisfies the equation $x = h + \frac{r}{60} = h + \frac{40}{60} = h + \frac{2}{3}$. The largest possible h is $h = 11$, so $x = 11 + \frac{2}{3} = \frac{35}{3}$.

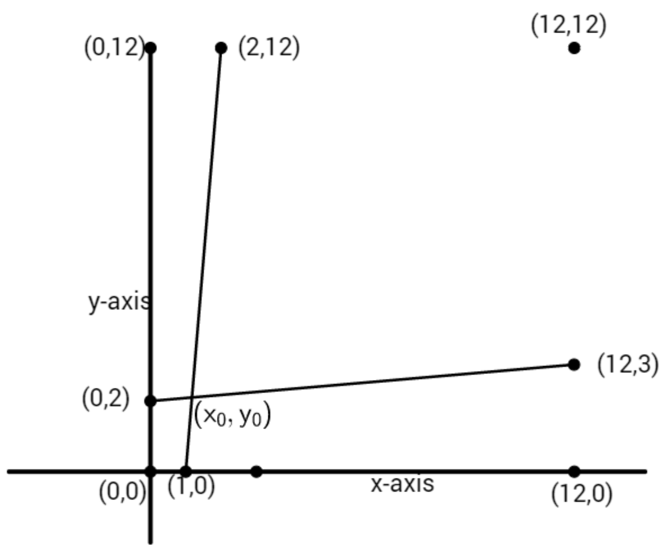
As mentioned before, there are 12 lines with slope of $m = 12$ that represent all the valid times of the clock, namely
$t = x = h + \frac{y}{12}$, $h = 0, 1, 2, \cdots, 11$, $0 \le y < 12$.
If we reflect the axes, (i.e. the $x - axis$ is the location of the M-hand and the $y - axis$ is the location of the H-hand), then all valid times will be the coordinates (x, y) that satisfy the equation $t = y = h + \frac{x}{12}$, $h = 0, 1, 2, \cdots, 11$, $0 \le x < 12$.

The figure below shows two lines: one of the 12 line without reflecting the axes, and one of the 12 lines when the axes are reflected. Also shown is the intersection point, (x_0, y_0), of these two lines. In general, the intersection point, (x_0, y_0), where $0 \leq x_0, y_0 < 12$, of any such two lines satisfies two equations:

1) $y_0 = h_1 + \frac{x_0}{12}, h_1 = 0, 1, 2, \cdots, 11,$

2) $x_0 = h_2 + \frac{y_0}{12}, h_2 = 0, 1, 2, \cdots, 11,$

and this figure is for the specific case: $h_1 = 2$, and $h_2 = 1$.



So, the point (x_0, y_0) of the previous figure simultaneously satisfies the following 2 equations:

1) $x = 1 + \frac{y}{12}, 0 \le y < 12$
2) $y = 2 + \frac{x}{12}, 0 \le x < 12$

Quiz 11:
What is the value of the x-coordinate, x_0, of the point (x_0, y_0) that simultaneously satisfies the following equations:

1) $x_0 = 1 + \frac{y_0}{12},$
2) $y_0 = 2 + \frac{x_0}{12},$

1. $\frac{13}{11}$

2. $\frac{150}{141}$

3. $\frac{168}{143}$

4. $\frac{169}{144}$

5. $\frac{18}{13}$

Answer to Quiz 11:
Multiply both equations by 12:

$$12x_0 = 12 + y_0,$$
$$12y_0 = 24 + x_0,$$

Group the equations in order of x_0, y_0, and the free terms:

$$y_0 - 12x_0 + 12 = 0,$$
$$12y_0 - x_0 - 24 = 0,$$

Multiply each of the terms of the first equation by 12:

$$12y_0 - 144x_0 + 144 = 0,$$
$$12y_0 - x_0 - 24 = 0,$$

Subtract the second equation from the first equation:

$$-143x_0 + 168 = 0,$$

Thus,

$$x_0 = \frac{168}{143},$$

From here one can calculate the value of y_0:

$$y_0 = 12x_0 - 12 = 12 \times \frac{168}{143} - 12 = 12 \times (\frac{168}{143} - 1) =$$
$$y_0 = 12 \times \frac{168 - 143}{143} = \frac{12 \times 25}{143} = \frac{300}{143}$$

Based on the discussion before, the point (x_0, y_0) is a valid time before and after swapping the axes, which simply means swapping the H-hand and the M-hand of a clock. The same is true for all intersection points (x_0, y_0), $0 \le x_0, y_0 < 12$, of the lines that simultaneously satisfy two equations:

1) $y_0 = h_1 + \frac{x_0}{12}, h_1 = 0, 1, 2, \cdots, 11,$

2) $x_0 = h_2 + \frac{y_0}{12}, h_2 = 0, 1, 2, \cdots, 11,$

How many such intersection points are there?
There are 144 intersection points of the lines. But one of the intersection points is $(12, 12)$ which is outside the defined range. So there are a total of 143 such valid times.