2021 workshop presentation (version of 20210223)

Part 1: Factoring of positive integers

A factor of a positive number N is a positive number that divides N.

For example, the numbers 1, 2, 3, and 6 are all the factors of the number 6.

We want to find a general answer to the following question: How many factors does a positive number N have?

First, consider some simple cases.

Quiz 1:
How many factors does the number 12 have?

1. **2**
2. **4**
3. **6**
4. **8**
5. **10**
6. **12**

Answer to Quiz 1:
$\mathbf{12 = 2 \times 2 \times 3.}$
It is easy to see that all the factors are:
1, 2, 3, 4, 6, and 12.
Thus, the number 12 has 6 factors.

Consider various specific values of $\boldsymbol{N}$.

Case 1: $\boldsymbol{N = p, p}$ is a prime number.
A prime number is a whole number greater than 1 that has no divisors other than itself and the number 1.

Thus, the factors of such $\boldsymbol{N}$ are the numbers 1 and $\boldsymbol{p}$ (total of 2 factors).

Example: $\boldsymbol{N = 11}$, the factors are 1 and 11.

Quiz 2:
Which of the following numbers is NOT a prime number?

1. **29**
2. **37**
3. **39**
4. **59**
5. **79**
6. **97**

Answer to Quiz 2:
A composite number greater than 1 is a number that is not a prime number.
It is clear that composite numbers less than the number 100 must have at least one prime factor which is less than 10.
Thus, all composite numbers smaller than 100 have at least one of the following prime factors: 2, 3, 5, or 7.
Thus, $39 = 3 \times 13$ is not a prime number.

Case 2: $N = p \times q$, p and q are different primes.
The factors of N are 1, p , q, and $p \times q$ (total of 4 factors).
Example: $N = 65 = 5 \times 13$, the factors of 65 are 1, 5, 13, and 65.

Quiz 3:
Which of the following numbers is in the form of $p \times q$ where p and q are different prime numbers?

1. **71**
2. **125**
3. **127**
4. **143**
5. **169**
6. **231**

Answer to Quiz 3:
71 and 127 are primes,
$\mathbf{125 = 5 \times 5 \times 5,}$
$\mathbf{143 = 11 \times 13}$ **(the correct answer),**
$\mathbf{169 = 13 \times 13,}$
$\mathbf{231 = 3 \times 7 \times 11.}$

Case 3: $\boldsymbol{N = p \times q \times q = p^1 \times q^2, p}$ **and** $\boldsymbol{q}$ **are different primes.**
The factors are $\boldsymbol{1 \times 1, 1 \times q, 1 \times q^2, p \times 1, p \times q, p \times q^2.}$
The total number of factors is 6.

Quiz 4:

Which of the following numbers is in the form of $p^1 \times q^2$ where p and q are different prime numbers?

1. **27**
2. **36**
3. **105**
4. **120**
5. **147**
6. **289**

Answer to Quiz 4:
$\mathbf{27 = 3 \times 3 \times 3.}$
$\mathbf{36 = 2 \times 2 \times 3 \times 3.}$
$\mathbf{105 = 3 \times 5 \times 7.}$
$\mathbf{120 = 2 \times 2 \times 2 \times 3 \times 5.}$
$\mathbf{147 = 3 \times 7 \times 7}$ **(the correct answer).**
$\mathbf{289 = 17 \times 17.}$

So, the number of factors of $\boldsymbol{N = p \times q^2}$ is:
$\mathbf{(1+1) \times (1+2) = 2 \times 3 = 6}$**, where 1 and 2 are the powers of $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively.**
Example: $\boldsymbol{N = 363 = 3 \times 121 = 3 \times 11 \times 11.}$
The 6 factors of 363 are 1, 11, 121, 3, 33, 363.
So, the total number of factors of 363 is:
$\mathbf{(1+1) \times (1+2) = 2 \times 3 = 6}$**, where 1 and 2 are the powers of 3 and 11 respectively.**
Or, more general, the number of factors of $\boldsymbol{N = p^n \times q^m}$ where $\boldsymbol{n}$ and $\boldsymbol{m}$ are whole numbers, and $\boldsymbol{p}$ and $\boldsymbol{q}$ are different primes is: $\boldsymbol{(n+1) \times (m+1)}$**.**

Quiz 5:

How many factors does the number 2000 have?

1. **12**
2. **15**
3. **16**
4. **20**
5. **24**
6. **25**

Answer to Quiz 5:
$\mathbf{2000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^4 \times 5^3.}$
So, the number of factors of 2000 is:
$\mathbf{(1+4) \times (1+3) = 5 \times 4 = 20.}$

Case 4 (the generalized case):
Based on the discussion above, below is a formula for the generalized case when a number N has only 3 different prime factors named p, q, and r.
The same method can be applied to any number N if we know ALL of its prime factors, and ALL their powers that divide N.
If $N = p^i \times q^j \times r^k$, and i, j, k are integers,
the number of factors of N is:
$\mathbf{(1+i) \times (1+j) \times (1+k).}$
Example: $N = 11^3 \times 13^5 \times 17^4$.
The number of factors of N is:
$\mathbf{(1+3) \times (1+5) \times (1+4) = 4 \times 6 \times 5 = 120.}$

Quiz 6:

How many factors does the number $N = 720$ have?

1. **20**
2. **24**
3. **30**
4. **60**
5. **72**

Answer to quiz 6:
$\boldsymbol{N = 720 = 5 \times 144 = 5 \times 9 \times 16 = 5^1 \times 3^2 \times 2^4}$**.**
Number of factors is:
$\boldsymbol{(1+1) \times (1+2) \times (1+4) = 2 \times 3 \times 5 = 30}$**.**

Quiz 7:

How many factors does the number $N = 3072$ have?

1. 14
2. 22
3. 24
4. 28
5. 44

Answer to quiz 7:
$N = 3072 = 3 \times 1024 = 3^1 \times 2^{10}$.
Number of factors is:
$(1+1) \times (1+10) = 2 \times 11 = 22$.

Consider a more complicated question about factors of a positive integer:
What is the sum of all the different factors of a number N?

To provide a general answer to this question we need to, first, study this for various simple cases of N.

Examples:

$N = 1$	**Factor is 1.**	**Sum of factors is 1.**
$N = 2$	**Factors are 1, 2.**	**Sum of factors is 3.**
$N = 3$	**Factors are 1, 3.**	**Sum of factors is 4.**

So, one can conclude that for $N = p$, p is prime, the sum of the factors is $1 + p$.

$N = 4 = 2 \times 2$ **Factors are 1, 2, and 4.**
Sum of the factors is $1 + 2 + 4 = 7$.
So, for $N = p^2$, (N is square of a prime number), one can conclude that the factors are 1, p, and p^2, and the sum of the factors is $1 + p + p^2$.

And, in general, for $N = p^n$, (power of a prime number), the sum of the factors is $1 + p + p^2 + \cdots + p^n$.

Quiz 8:
What is the sum of all the factors of 81?
1. 81
2. 82
3. 121
4. 244
5. 6643

Answer to quiz 8:
$\mathbf{81 = 3 \times 3 \times 3 \times 3 = 3^4}$.
So, the sum of the factors is:
$\mathbf{1 + 3^1 + 3^2 + 3^3 + 3^4 = 1 + 3 + 9 + 27 + 81 = 121}$.

More examples:
$\boldsymbol{N = 6 = 2 \times 3}$ **Factors are 1, 3, 2, 6.**
Or, more specifically: $\mathbf{1 \times 1, 1 \times 3, 2 \times 1, 2 \times 3}$.
Or, if we group the factors, the sum of factors is:
$\mathbf{(1 + 2) \times (1 + 3) = 3 \times 4 = 12}$.

So, for $\boldsymbol{N = p \times q}$**,** $(\boldsymbol{p, q}$ **are prime numbers), one can conclude that the sum of the factors is:** $\boldsymbol{(1 + p) \times (1 + q)}$.

And, more generalized, if $\boldsymbol{N = p^n \times q^m}$**, where** $\boldsymbol{p}$ **and** $\boldsymbol{q}$ **are different primes, the sum of the factors is:**
$\boldsymbol{(1 + p + p^2 + \cdots + p^n) \times (1 + q + q^2 + \cdots + q^m)}$.

And, for the most generalized case when we know ALL the prime factors of a number $\boldsymbol{N}$ **along with ALL of their respective powers that divide** $\boldsymbol{N}$**, we can generalize the formula above.**

Example: $\boldsymbol{N = 504 = 2^3 \times 3^2 \times 7^1}$.
The sum of all the factors is:
$\mathbf{(1 + 2 + 2^2 + 2^3) \times (1 + 3 + 3^2) \times (1 + 7) = 15 \times 13 \times 8}$.
So, the sum of all the factors of 504 is 1560.

Quiz 9:

What is sum of all the factors of $N = 600$?

1. 1224
2. 1860
3. 2325
4. 2480
5. 2880

Answer to Quiz 9

$\boldsymbol{N = 600 = 2 \times 100 \times 3 = 2^3 \times 3 \times 5^2.}$

Thus, the sum of all factors of 600 is:

$\boldsymbol{(1 + 2 + 2^2 + 2^3) \times (1 + 3) \times (1 + 5 + 5^2) = 15 \times 4 \times 31}$

$\boldsymbol{60 \times 31 = 1860.}$

Quiz 10 (2 parts):

Quiz 10a:

How many numbers are there such that the sum of all their factors is 72?

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6

Quiz 10b:

What is the smallest number *N* whose sum of all factors is 72?

1. 24
2. 28
3. 30
4. 46
5. 51

Answer to Quizzes 10a and 10b:
This a much more complicated question than all of the questions in the previous quizzes.
To solve this problem, we need to use the various formats of the formula of the sum of all factors, and find all possible numbers, N, that provide answer of 72 for the sum of all their factors.
So, assume that the answer is 72, and find possible candidates for N:
Case a)
We know that the largest N that satisfies the condition is the number 71 since 71 is a prime and since $71 + 1 = 72$.

In addition, since 72 is a composite number, we need to investigate all possible acceptable multiplications of 2 or more factors of 72, ($72 = 2^3 \times 3^2$), with the condition that each factor is greater than 2. This is because the smallest number that divides a composite number N is 2, and it corresponds to the number $1 + 2$, which must be a factor of 72.

Check valid possible ways of representing 72 as a multiplication of 2 numbers greater than 2:
Case b):
$72 = 3 \times 24 = (1 + 2) \times (1 + 23)$
Both 2 and 23 are primes, so one possible candidate is:
$46 = 2 \times 23.$
Could it be that $N = 2 \times p^k$, where $p > 2$, and the number 24 satisfies: $24 = 1 + p + p^2 + \cdots + p^k$?
The only possible candidate for p is the number 3,
and for $p = 3$, $1 + 3 + 9 \neq 24.$
Thus, there are no other solutions for the case of $3 \times 24 = 72.$

Case c):
$72 = 4 \times 18 = (1 + 3) \times (1 + 17)$
Both 3 and 17 are primes, so, one possible candidate is:
$51 = 3 \times 17.$
Could it be that $N = 3 \times p^k$, where $p \neq 3$, and the number 18 satisfies: $18 = 1 + p + p^2 + \cdots + p^k$?
The only possible candidate for p is the number 2,
and for $p = 2$, $1 + 2 + 4 + 8 \neq 18.$
Thus, there are no other solutions for the case of $4 \times 18 = 72.$

Case d): $\mathbf{6 \times 12 = (1+5) \times (1+11)}$
So, one possible candidate is $\mathbf{55 = 5 \times 11.}$
Could there be other options where the sum of 3 or more powers of a single prime adds to 12:
For $\boldsymbol{p = 2, 1+2+4 \neq 12.}$
For $\boldsymbol{p = 3,\ 1+3+9 \neq 12.}$
Also, the number 6 can not be the sum of 3 or more powers of of a single prime.
Thus, there are no other solutions for $\mathbf{6 \times 12 = 72.}$

Case e)
$\mathbf{8 \times 9 = (1+7) \times (1+8)}$
7 is a prime number. But 8 is not a prime number, and, also, there is no other option where the sum of 3 or more powers of a single prime adds to 9:
$\mathbf{1+2+4 \neq 9.}$
So, $\mathbf{8 \times 9 = 72}$ is no good

Check valid possible ways of representing 72 as a multiplication of 3 numbers, where each number is greater than 2:

Case f)
$\mathbf{3 \times 3 \times 8 = (1+2) \times (1+2) \times (1+7)}$. **This option is no good because the multiplication is not in an acceptable format that is required by the formula that we developed earlier (the number 3 is allowed as a factor only once).**

Case g)
$\mathbf{3 \times 4 \times 6 = (1+2) \times (1+3) \times (1+5)}$.
2, 3, and 5 are primes, so, we find another possible candidate, $\mathbf{30 = 2 \times 3 \times 5}$.

Summary of all found candidates for $\boldsymbol{N}$ **in a)-g):**
71, 46, 51, 55, 30
So the answer to the quiz 10a is 5, and the answer to quiz 10b is: $\boldsymbol{N = 30}$.